

Stellar Equilibrium and Gravitational Collapse in the Nonsymmetric Gravitational Theory

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(February 1, 2008)

Abstract

We establish the formalism in the nonsymmetric gravitational theory (NGT) for stellar equilibrium and gravitational collapse. We study the collapse of a pressureless, spherically symmetric dust cloud. By assuming that the interior solution is smoothly matched at the surface of the star to the quasi-static, spherically symmetric vacuum solution, we find that the star does not collapse to a black hole. It is anticipated that the final collapsed object will reach a state of equilibrium, and will emit thermal, gravitational and other forms of radiation, although the radiation may be emitted only in small amounts if the red shift from the surface of the compact object is large. No Hawking radiation is emitted and the information loss problem can be resolved at the classical level.

From its *seeming* to me – or to everyone – to do so, it doesn't follow that it is so. What we can ask is whether it can make sense to doubt it.

L. Wittgenstein, *On Certainty*

I. INTRODUCTION

Recently, a new perturbatively consistent version of the nonsymmetric gravitational theory (NGT) has been formulated [1–4]. The linear approximation yields a theory free of ghost poles and tachyons and a Hamiltonian that is bounded from below [1,2]. The flux of gravitational energy at asymptotic infinity is positive definite for a finite value of the range parameter $a = \mu^{-1}$. An analysis of spherically symmetric systems in the new NGT has been carried out by Clayton [5].

It was conjectured that on the basis of a static spherically symmetric vacuum solution of the NGT field equations, in the long-range approximation, $a \gg 2M$, no black holes would form during the collapse of a star with a mass greater than the Chandrasekhar mass limit [6,7]. In order to establish the correctness of this conjecture, it is necessary to study the situation of the physical collapse of a star containing matter as the star's pressure falls to zero.

In order to make the solving of the field equations manageable, we must make some simplifying approximations. We assume that the direct coupling to the skew part of the nonsymmetric source tensor is small and can be neglected during the collapse. We also make some simplifying approximations about the nature of the skew symmetric contributions to the field equations. With these approximations, we find that if the interior solution of the star and the exterior quasi-static solution are matched, the star collapses without forming an event horizon (black hole) and that the collapse can stop at some time, $t = t_0$, before it forms a state of singular energy density.

The static vacuum (Wyman) solution contains no event horizons and the non-Riemannian geometry is singularity-free in physical spacetime. The NGT does not possess a Birkhoff theorem [2,5], but it is reasonable to suppose that the final state of collapse will, under physically realistic conditions, tend asymptotically with time towards a pseudo-stationary final state. This state is one in which spacetime is invariant under an isometry group generated by a Killing vector field which is timelike for $0 < r < \infty$. One anticipates

that nonstationary motions will in general be damped out by gravitational radiation, viscosity, etc. If such a stationary, massive final collapsed object forms, it will for practical purposes be indistinguishable from the observed massive “black holes” reported abundantly in the literature. The final collapsed object in NGT will radiate thermal radiation, gravitational radiation, etc., but not Hawking radiation, since the latter is a unique feature of a black hole event horizon. This could eliminate the problem of information loss associated with black holes in GR [3,8].

We begin, in Section II, with a presentation of the NGT field equations and the conservation laws. Then, in Section III, we provide a derivation of the motion of test particles, based on the fluid energy momentum tensor. In Section IV, we study the formalism for a spherically symmetric system and derive the time dependent field equations for such a system. In Sections V and VI, we develop the formalism for stellar equilibrium and gravitational collapse and in Section VII, we carry out an analysis of the field equations for the gravitational collapse of a spherically symmetric, pressureless dust cloud. This model of collapse is the equivalent, in NGT, of the Oppenheimer-Snyder collapse model in GR [9]. In Section VIII, we study the static spherically symmetric vacuum solution, and discuss the dual roles of the Riemannian and non-Riemannian geometries in NGT. In Section IX, we study the matching of the interior and the exterior metrics during collapse. Finally, in Section X, we summarize the results of this paper and discuss the claim made by Burko and Ori [10,11] that black holes are anticipated in NGT collapse, using the linear approximation of an expansion of the NGT field equations about the Schwarzschild solution of GR.

II. STRUCTURE OF THE NONSYMMETRIC GRAVITATIONAL THEORY

The NGT is a geometric theory of gravity based on a nonsymmetric field structure with a nonsymmetric fundamental tensor $g_{\mu\nu}$, defined by

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}. \quad (1)$$

The affine connection coefficients, $\Gamma_{\mu\nu}^\lambda$, are also nonsymmetric:

$$\Gamma_{\mu\nu}^\lambda = \Gamma_{(\mu\nu)}^\lambda + \Gamma_{[\mu\nu]}^\lambda. \quad (2)$$

We define the inverse tensor $g^{\mu\nu}$ by the relation

$$g^{\mu\nu} g_{\mu\alpha} = g^{\nu\mu} g_{\alpha\mu} = \delta_\alpha^\nu. \quad (3)$$

The NGT Ricci curvature tensor $R_{\mu\nu}(W)$ is given by

$$R_{\mu\nu}(W) = W_{\mu\nu,\beta}^\beta - \frac{1}{2}(W_{\mu\beta,\nu}^\beta + W_{\nu\beta,\mu}^\beta) - W_{\alpha\nu}^\beta W_{\mu\beta}^\alpha + W_{\alpha\beta}^\beta W_{\mu\nu}^\alpha, \quad (4)$$

where the $W_{\mu\nu}^\lambda$ are the unconstrained nonsymmetric connection coefficients, defined in terms of the affine connection coefficients through the relation:

$$W_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \frac{2}{3}\delta_\mu^\lambda W_\nu, \quad (5)$$

where $W_\mu = W_{[\mu\alpha]}^\alpha$. It follows from (5) that $\Gamma_\mu = \Gamma_{[\mu\lambda]}^\lambda = 0$. The NGT Ricci scalar is given by $R(W) = g^{\mu\nu} R_{\mu\nu}(W)$.

The NGT field equations take the form:

$$G_{\mu\nu}(W) + \Lambda g_{\mu\nu} + S_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (6)$$

$$\mathbf{g}^{[\mu\nu]},_{,\nu} = -\frac{1}{2}\mathbf{g}^{(\mu\sigma)}W_\sigma, \quad (7)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R,$$

and

$$S_{\mu\nu} = \frac{1}{4}\mu^2 C_{\mu\nu} - \frac{1}{6}P_{\mu\nu}^*.$$

Moreover,

$$C_{\mu\nu} = \frac{1}{2}g_{\mu\nu}g^{[\alpha\beta]}g_{[\beta\alpha]} + g^{[\alpha\beta]}g_{\mu\alpha}g_{\beta\nu} + g_{[\mu\nu]},$$

and $P_{\mu\nu}^* = P_{\mu\nu} - \frac{1}{2}g_{\mu\nu}P$ with $P_{\mu\nu} = W_\mu W_\nu$. Λ and μ denote the cosmological constant and a “mass” parameter associated with the antisymmetric field $g_{[\mu\nu]}$, respectively.

We can write the field equations (6) in the form:

$$R_{\mu\nu}(W) = \Lambda g_{\mu\nu} + 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T - \frac{1}{32\pi}\mu^2 I_{\mu\nu} + \frac{1}{48\pi}P_{\mu\nu}), \quad (8)$$

and $T = g^{\mu\nu}T_{\mu\nu}$. Also, we have

$$I_{\mu\nu} = C_{\mu\nu} - \frac{1}{2}g_{\mu\nu}C = g^{[\alpha\beta]}g_{\mu\alpha}g_{\beta\nu} + \frac{1}{2}g_{\mu\nu}g^{[\alpha\beta]}g_{[\alpha\beta]} + g_{[\mu\nu]}, \quad (9)$$

where $C = g^{\mu\nu}C_{\mu\nu}$.

In empty space, the field equations (8) become:

$$R_{\mu\nu}(\Gamma) = \frac{2}{3}W_{[\nu,\mu]} + \Lambda g_{\mu\nu} - \frac{1}{4}\mu^2 I_{\mu\nu} + \frac{1}{6}P_{\mu\nu}, \quad (10)$$

where

$$R_{\mu\nu}(\Gamma) = \Gamma_{\mu\nu,\beta}^\beta - \frac{1}{2}(\Gamma_{(\mu\beta),\nu}^\beta + \Gamma_{(\nu\beta),\mu}^\beta) - \Gamma_{\alpha\nu}^\beta \Gamma_{\mu\beta}^\alpha + \Gamma_{\mu\nu}^\alpha \Gamma_{(\alpha\beta)}^\beta.$$

From the variational principle and the general covariance of the Lagrangian density, we can obtain the four Bianchi identities:

$$\frac{1}{2}(G_{\rho\nu}(\Gamma)\mathbf{g}^{\sigma\nu} + G_{\nu\rho}(\Gamma)\mathbf{g}^{\nu\sigma})_{,\sigma} + \frac{1}{2}\mathbf{G}_{\tau\nu}(\Gamma)g^{\tau\nu}_{,\rho} = 0. \quad (11)$$

The Bianchi identities lead to the four conservation laws [4]:

$$g_{\mu\lambda}\mathbf{T}^{\mu\rho}_{,\rho} + g_{\lambda\mu}\mathbf{T}^{\rho\mu}_{,\rho} + 2[\mu\nu, \lambda]\mathbf{T}^{\mu\nu} = 0, \quad (12)$$

where

$$[\mu\nu, \lambda] = \frac{1}{2}(g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}).$$

This is known as the generalized law of energy-momentum conservation in NGT.

III. EQUATIONS OF MOTION OF TEST PARTICLES

Let us consider the equations of motion of test particles derived from the conservation law (12). We shall assume that the particle is confined to a tube Σ , whose linear cross section

dimensions are small compared to the length characterizing the gradient of the background metric. The fundamental tensor $g_{\mu\nu}$ consists of a piece $g_{\mu\nu}^{(0)}$ corresponding to the continuous field at points along the world line of the test particle, and the part $\delta g_{\mu\nu}$ that describes the correction to the background $g_{\mu\nu}^{(0)}$ field due to the test particle [12]. We can then write

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}. \quad (13)$$

To the first order of approximation, we keep in the field equations only the terms which are linear in $\delta g_{\mu\nu}$. The energy-momentum tensor associated with the test particle is $\delta T^{\mu\nu}$. It is assumed that the energy-momentum tensor $T^{(0)\mu\nu}$, associated with the background metric $g_{\mu\nu}^{(0)}$, vanishes inside as well as near the test particle. We adopt the notation:

$$\mathcal{T}^{\mu\nu} = \sqrt{-g} \delta T^{\mu\nu}. \quad (14)$$

The energy-momentum tensor for a fluid is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} + K^{[\mu\nu]}, \quad (15)$$

where $u^\mu = dx^\mu/ds$ is the four-velocity of a fluid element, normalized so that

$$g_{\mu\nu}u^\mu u^\nu = g_{(\mu\nu)}u^\mu u^\nu = 1. \quad (16)$$

In the test particle limit, $p \rightarrow 0$ and we get

$$\delta T^{[\mu\nu]} = K^{[\mu\nu]}.$$

The equations of motion of the test particle are then given by

$$\mathcal{T}^{(\mu\nu)}_{,\nu} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}^{(0)} \mathcal{T}^{(\alpha\beta)} = \mathbf{f}^\mu, \quad (17)$$

where

$$f^\mu = s^{(0)\mu\alpha} (g_{[\beta\alpha]}^{(0)} K^{(0)[\beta\nu]}_{,\nu} + K^{(0)[\rho\nu]} [[\rho\nu], \alpha]^{(0)}), \quad (18)$$

and we have used the inverse symmetric tensor $s^{\mu\nu}$ defined by

$$s^{\mu\nu} g_{(\sigma\nu)} = \delta_\sigma^\mu$$

to raise suffixes. Moreover,

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} s^{\lambda\rho} (s_{\mu\rho,\nu} + s_{\rho\nu,\mu} - s_{\mu\nu,\rho}). \quad (19)$$

Let us define the proper mass of the test particle by the equation

$$\int d^3x \mathcal{T}^{00} = u^0 \int d^3x \sqrt{-g} \rho_0 = m_t \frac{dt}{d\tau},$$

where ρ_0 is the proper mass density, m_t denotes the test particle mass and $d\tau$ is the proper time along the world line. The equation of motion now takes the form:

$$\frac{d}{d\tau}(m_t u^\mu) + m_t \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}^{(0)} u^\alpha u^\beta = m_t \tilde{f}^\mu, \quad (20)$$

where we have defined

$$f^\mu = m_t \tilde{f}^\mu.$$

We have

$$\frac{d}{d\tau}(m_t u^\mu) = m_t \frac{du^\mu}{d\tau} + \frac{dm_t}{d\tau} u^\mu.$$

From the condition:

$$g_{(\mu\nu)}^{(0)} u^\mu u^\nu = 1,$$

we get

$$\frac{1}{2} (g_{(\mu\nu)}^{(0)} u^\mu u^\nu)_{|\sigma} = g_{(\mu\nu)}^{(0)} u^\nu u^\mu_{|\sigma} = u_\mu u^\mu_{|\sigma} = 0,$$

where $|$ denotes covariant differentiation with respect to the Christoffel symbol. We now have

$$u_\lambda \left(\frac{du^\lambda}{d\tau} + \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}^{(0)} u^\alpha u^\beta \right) = u_\lambda u^\lambda_{|\mu} u^\mu = 0.$$

Multiplying (20) by u_μ , we get

$$\frac{dm_t}{d\tau} = m_t u_\mu \tilde{f}^\mu.$$

We now obtain the equation of motion of the test particle (we drop the (0) notation):

$$\frac{du^\mu}{d\tau} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} u^\alpha u^\beta = \tilde{f}^\mu - u_\alpha \tilde{f}^\alpha u^\mu. \quad (21)$$

We see that, depending on the model chosen for $K^{[\mu\nu]}$, there could be a contribution due to the non-conservation of the mass, caused by an exchange of energy with the skew field $g_{[\mu\nu]}$.

According to Eq.(21), test particles fall in an NGT gravitational field independently of their composition, so the weak equivalence principle is satisfied in the new version of NGT. However, the strong equivalence principle is not satisfied in the theory, because the non-gravitational laws of physics are not the same in different locally Minkowskian frames of reference; the skew part of the connection $\Gamma_{[\mu\nu]}^\lambda$ is a tensor which cannot be transformed away at a point, in contrast to the Christoffel connection $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$.

If we assume that $K^{[\mu\nu]}$ is small and can be neglected, we obtain the geodesic equation of motion for test particles [13]:

$$\frac{du^\mu}{d\tau} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} u^\alpha u^\beta = 0. \quad (22)$$

IV. THE FIELD EQUATIONS FOR A SPHERICALLY SYMMETRIC SYSTEM

For the case of a spherically symmetric field, the canonical form of $g_{\mu\nu}$ in NGT is given by

$$g_{\mu\nu} = \begin{pmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & f \sin\theta & 0 \\ 0 & -f \sin\theta & -\beta \sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{pmatrix},$$

where α, β, γ and w are functions of r and t . The tensor $g^{\mu\nu}$ has the components:

$$g^{\mu\nu} = \begin{pmatrix} \frac{\gamma}{w^2 - \alpha\gamma} & 0 & 0 & \frac{w}{w^2 - \alpha\gamma} \\ 0 & -\frac{\beta}{\beta^2 + f^2} & \frac{f \text{CSC}\theta}{\beta^2 + f^2} & 0 \\ 0 & -\frac{f \text{CSC}\theta}{\beta^2 + f^2} & -\frac{\beta \text{CSC}^2\theta}{\beta^2 + f^2} & 0 \\ -\frac{w}{w^2 - \alpha\gamma} & 0 & 0 & -\frac{\alpha}{w^2 - \alpha\gamma} \end{pmatrix}.$$

We shall assume that $w = 0$ and only the $g_{[23]}$ component of $g_{[\mu\nu]}$ is different from zero. It can be proved that only the static solution for $g_{[23]}$ satisfies the physical, asymptotically flat boundary conditions [5,14].

We have

$$\sqrt{-g} = \sin\theta[(\alpha\gamma - w^2)(\beta^2 + f^2)]^{1/2}.$$

The vector W_μ can be determined from:

$$W_\mu = -\frac{2}{\sqrt{-g}}s_{\mu\rho}\mathbf{g}^{[\rho\sigma]}_{,\sigma}. \quad (23)$$

For the spherically symmetric field with $w = 0$ it follows from (23) that $W_\mu = 0$. The field equations (6) and (7) for the spherically symmetric system now take the simpler form:

$$G_{\mu\nu}(\Gamma) + \Lambda g_{\mu\nu} + \frac{1}{4}\mu^2 C_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (24)$$

$$\mathbf{g}^{[\mu\nu]}_{,\nu} = 0. \quad (25)$$

Let us now reexpress the conservation laws in a form suitable for calculations in a spherically symmetric system. Using the Einstein notation, the compatibility equations are given by

$$\mathbf{g}^{\mu+\nu-}_{;\sigma} \equiv \mathbf{g}^{\mu\nu}_{,\sigma} + \mathbf{g}^{\rho\nu}\Gamma_{\rho\sigma}^\mu + \mathbf{g}^{\mu\rho}\Gamma_{\sigma\rho}^\nu - \mathbf{g}^{\mu\nu}\Gamma_{(\sigma\alpha)}^\alpha = 0, \quad (26)$$

where

$$A^{\mu+}_{;\sigma} = A^\mu_{,\sigma} + A^\rho\Gamma_{\rho\sigma}^\mu,$$

and

$$A^{\mu-}_{;\sigma} = A^\mu_{,\sigma} + A^\rho\Gamma_{\sigma\rho}^\mu.$$

We have

$$\text{Re}[(G_{\rho-\nu-}(\Gamma)\mathbf{g}^{\sigma+\nu-})_{;\sigma}] = \frac{1}{2}[(G_{\rho-\nu-}(\Gamma)\mathbf{g}^{\sigma+\nu-})_{;\sigma} + (G_{\nu+\rho+}(\Gamma)\mathbf{g}^{\nu+\sigma-})_{;\sigma}] = 0. \quad (27)$$

This can be written as

$$\text{Re}[(G_{\rho-\nu-}(\Gamma)\mathbf{g}^{\sigma+\nu-})_{;\sigma}] = \text{Re}[(G_{\rho\nu}(\Gamma)\mathbf{g}^{\sigma\nu})_{,\sigma} - \frac{1}{2}G_{\tau\nu}(\Gamma)(\mathbf{g}^{\sigma\nu}\Gamma_{\sigma\rho}^{\tau} + \mathbf{g}^{\tau\sigma}\Gamma_{\rho\sigma}^{\nu})] = 0, \quad (28)$$

which leads to the result:

$$\text{Re}[(G_{\rho-\nu-}(\Gamma)\mathbf{g}^{\sigma+\nu-})_{;\sigma}] = \frac{1}{2}(G_{\rho\nu}(\Gamma)\mathbf{g}^{\sigma\nu})_{,\sigma} + \frac{1}{2}(G_{\nu\rho}(\Gamma)\mathbf{g}^{\nu\sigma})_{,\sigma} + \frac{1}{2}\mathbf{G}_{\tau\nu}(\Gamma)g^{\tau\nu}_{,\rho} = 0. \quad (29)$$

This is the same as the Bianchi identities (11). From (24), we have for the spherically symmetric conservation laws:

$$\text{Re}[(T_{\rho-\nu-}\mathbf{g}^{\sigma+\nu-})_{;\sigma}] = 0. \quad (30)$$

For a comoving coordinate system, we obtain

$$u^0 = \frac{1}{\sqrt{\gamma}}, \quad u^r = u^\theta = u^\phi = 0. \quad (31)$$

We define

$$T_{\mu\nu} = g_{\mu\beta}g_{\alpha\nu}T^{\alpha\beta}, \quad (32)$$

which is a Hermitian symmetric tensor, $T_{\mu\nu} = \tilde{T}_{\nu\mu}$, when $g_{\mu\nu}$ is defined to be Hermitian symmetric: $g_{\mu\nu} = \tilde{g}_{\nu\mu}$. From (3) and (15) we get

$$T = \rho - 3p + g_{[\alpha\beta]}K^{[\alpha\beta]} = \rho - 3p + 2fK,$$

where we have defined: $K^{[23]} = K/\sin\theta$.

The NGT field equations in the presence of sources are given by

$$\begin{aligned} R_{11}(\Gamma) &= -\frac{1}{2}A'' - \frac{1}{8}[(A')^2 + 4B^2] + \frac{\alpha'A'}{4\alpha} + \frac{\gamma'}{2\gamma}\left(\frac{\alpha'}{2\alpha} - \frac{\gamma'}{2\gamma}\right) \\ &\quad - \left(\frac{\gamma'}{2\gamma}\right)' + \frac{\partial}{\partial t}\left(\frac{\dot{\alpha}}{2\gamma}\right) + \frac{\dot{\alpha}}{2\gamma}\left(\frac{\dot{\gamma}}{2\gamma} - \frac{\dot{\alpha}}{2\alpha} + \frac{1}{2}\dot{A}\right) \\ &\quad + \Lambda\alpha - \frac{1}{4}\mu^2\frac{\alpha f^2}{\beta^2 + f^2} \\ &= 4\pi\alpha(\rho - p + 2fK), \end{aligned} \quad (33a)$$

$$R_{22}(\Gamma) = R_{33}(\Gamma)\text{cosec}^2\theta = 1 + \left(\frac{2fB - \beta A'}{4\alpha}\right)' + \left(\frac{2fB - \beta A'}{8\alpha^2\gamma}\right)(\alpha'\gamma + \gamma'\alpha)$$

$$\begin{aligned}
& + \frac{B(fA' + 2\beta B)}{4\alpha} - \frac{\partial}{\partial t} \left(\frac{2fD - \beta\dot{A}}{4\gamma} \right) - \frac{2fD - \beta\dot{A}}{8\alpha\gamma^2} (\dot{\alpha}\gamma + \dot{\gamma}\alpha) \\
& - \frac{D}{4\gamma} (f\dot{A} + 2\beta D) + \Lambda\beta + \frac{1}{4}\mu^2 \frac{\beta f^2}{\beta^2 + f^2} \\
& = 4\pi\beta(\rho - p + 2fK),
\end{aligned} \tag{33b}$$

$$\begin{aligned}
R_{00}(\Gamma) &= -\frac{1}{2}\ddot{A} - \frac{1}{8}(\dot{A}^2 + 4D^2) + \frac{\dot{\gamma}}{4\gamma}\dot{A} + \frac{\dot{\alpha}}{2\alpha} \left(\frac{\dot{\gamma}}{2\gamma} - \frac{\dot{\alpha}}{2\alpha} \right) \\
& - \frac{\partial}{\partial t} \left(\frac{\dot{\alpha}}{2\alpha} \right) + \left(\frac{\gamma'}{2\alpha} \right)' + \frac{\gamma'}{2\alpha} \left(\frac{\alpha'}{2\alpha} - \frac{\gamma'}{2\gamma} + \frac{1}{2}A' \right) \\
& - \Lambda\gamma + \frac{1}{4}\mu^2 \frac{\gamma f^2}{\beta^2 + f^2} \\
& = 4\pi\gamma(\rho + 3p - 2fK),
\end{aligned} \tag{33c}$$

$$R_{[10]}(\Gamma) = 0, \tag{33d}$$

$$\begin{aligned}
R_{(10)}(\Gamma) &= -\frac{1}{2}\dot{A}' + \frac{1}{4}A' \left(\frac{\dot{\alpha}}{\alpha} - \frac{1}{2}\dot{A} \right) + \frac{1}{4} \frac{\gamma'\dot{A}}{\gamma} - \frac{1}{2}BD \\
& = 0,
\end{aligned} \tag{33e}$$

$$\begin{aligned}
R_{[23]}(\Gamma) &= \sin\theta \left[\left(\frac{fA' + 2\beta B}{4\alpha} \right)' + \frac{1}{8\alpha} (fA' + 2\beta B) \left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} \right) \right. \\
& - \frac{B}{4\alpha} (2fB - \beta A') - \frac{1}{8\gamma} (f\dot{A} + 2\beta D) \left(\frac{\dot{\gamma}}{\gamma} + \frac{\dot{\alpha}}{\alpha} \right) \\
& - \frac{\partial}{\partial t} \left(\frac{f\dot{A} + 2\beta D}{4\gamma} \right) + \frac{D}{4\gamma} (2fD - \beta\dot{A}) \Big] \\
& - \left[\Lambda f - \frac{1}{4}\mu^2 f \left(1 + \frac{\beta^2}{\beta^2 + f^2} \right) \right] \sin\theta \\
& = -4\pi f \sin\theta (\rho - p).
\end{aligned} \tag{33f}$$

Here, prime denotes differentiation with respect to r , $\dot{A} = \partial A / \partial t$, and we have used the notation:

$$A = \ln(\beta^2 + f^2), \tag{34a}$$

$$B = \frac{f\beta' - \beta f'}{\beta^2 + f^2}, \tag{34b}$$

$$D = \frac{\dot{\beta}f - \dot{f}\beta}{\beta^2 + f^2}. \tag{34c}$$

V. STATIC EQUILIBRIUM EQUATIONS

Let us consider the static hydrodynamic equations for stellar equilibrium. For a static spherically symmetric system $g_{\mu\nu}, p, \rho$ and $K^{[\mu\nu]}$ are functions only of the radial coordinate r . For a fluid at rest, Eq.(31) determines the velocity components. To simplify the field equations, let us assume that the direct coupling of $g_{[23]}$ to $K^{[23]}$ is small and can be neglected. We also adopt the long-range approximation $\mu \approx 0$ and we choose $\Lambda = 0$. The field equations are given by

$$\begin{aligned} R_{11}(\Gamma) &= -\frac{1}{2}A'' - \frac{1}{8}[(A')^2 + 4B^2] + \frac{\alpha'A'}{4\alpha} + \frac{\gamma'}{2\gamma}\left(\frac{\alpha'}{2\alpha} - \frac{\gamma'}{2\gamma}\right) \\ &= 4\pi\alpha(\rho - p), \end{aligned} \tag{35a}$$

$$\begin{aligned} R_{22}(\Gamma) = R_{33}(\Gamma)\text{cosec}^2\theta &= 1 + \left(\frac{2fB - \beta A'}{4\alpha}\right)' + \left(\frac{2fB - \beta A'}{8\alpha^2\gamma}\right)(\alpha'\gamma + \gamma'\alpha) \\ &\quad + \frac{B(fA' + 2\beta B)}{4\alpha} \\ &= 4\pi\beta(\rho - p), \end{aligned} \tag{35b}$$

$$\begin{aligned} R_{00}(\Gamma) &= \left(\frac{\gamma'}{2\alpha}\right)' + \frac{\gamma'}{2\alpha}\left(\frac{\alpha'}{2\alpha} - \frac{\gamma'}{2\gamma} + \frac{1}{2}A'\right) \\ &= 4\pi\gamma(\rho + 3p), \end{aligned} \tag{35c}$$

$$R_{[10]}(\Gamma) = 0, \tag{35d}$$

$$R_{(10)}(\Gamma) = 0, \tag{35e}$$

$$\begin{aligned} R_{[23]}(\Gamma) &= \sin\theta \left[\left(\frac{fA' + 2\beta B}{4\alpha}\right)' + \frac{1}{8\alpha}(fA' + 2\beta B)\left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right) - \frac{B}{4\alpha}(2fB - \beta A') \right] \\ &= -4\pi f \sin\theta(\rho - p). \end{aligned} \tag{35f}$$

The conservation laws (30) give

$$p' = -\frac{1}{2}(\rho + p)(\ln \gamma)'. \tag{36}$$

We can further simplify the equations by adopting two approximation schemes:

$$r^2 \gg f(r), \tag{37}$$

and

$$f(r) \gg r^2, \quad (38)$$

where we have chosen $\beta(r) = r^2$. For the first approximation scheme, we obtain for the $11/\alpha, 22/\beta, 00/\gamma$ and $[23]/f$ field equations:

$$-\frac{\gamma''}{2\alpha\gamma} + \frac{\gamma'}{4\alpha\gamma} \left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} \right) + \frac{\alpha'}{\alpha^2 r} + I = 4\pi(\rho - p), \quad (39a)$$

$$\frac{1}{r^2} - \frac{1}{\alpha r^2} + \frac{1}{2r\alpha} \left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} \right) + N = 4\pi(\rho - p), \quad (39b)$$

$$\frac{\gamma''}{2\alpha\gamma} - \frac{\gamma'}{4\alpha\gamma} \left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} \right) + \frac{\gamma'}{r\alpha\gamma} + L = 4\pi(\rho + 3p), \quad (39c)$$

$$-\frac{\alpha' f'}{4\alpha^2 f} + \frac{\gamma' f'}{4\alpha\gamma f} - \frac{\gamma'}{r\alpha\gamma} + \frac{f''}{2\alpha f} - \frac{f'}{r\alpha f} + \frac{\alpha'}{r\alpha^2} = 4\pi(\rho - p). \quad (39d)$$

Here, we have defined

$$I(r) = -\frac{\alpha' f^2}{\alpha^2 r^5} + \frac{8ff'}{\alpha r^5} - \frac{8f^2}{\alpha r^6} - \frac{3f'^2}{2\alpha r^4} + \frac{\alpha' f f'}{2\alpha^2 r^4} - \frac{f f''}{\alpha r^4}, \quad (40a)$$

$$N(r) = -\frac{\alpha' f^2}{\alpha^2 r^5} - \frac{2f^2}{\alpha r^6} + \frac{\gamma' f^2}{\alpha\gamma r^5} + \frac{\alpha' f f'}{2\alpha^2 r^4} - \frac{f f''}{\alpha r^4} + \frac{3f f'}{\alpha r^5} - \frac{f'^2}{2\alpha r^4} - \frac{\gamma' f f'}{2\alpha\gamma r^4}, \quad (40b)$$

$$L(r) = \frac{\gamma' f f'}{2\alpha\gamma r^4} - \frac{\gamma' f^2}{\alpha\gamma r^5}. \quad (40c)$$

Let us now consider the following combination: $11/2\alpha + 22/\beta + 00/2\gamma$:

$$\frac{\alpha'}{r\alpha^2} + \frac{1}{r^2} - \frac{1}{\alpha r^2} + G = 8\pi\rho, \quad (41)$$

where

$$G(r) = \frac{1}{2}[I(r) + L(r)] + N(r).$$

Eq.(41) can be written:

$$\left(\frac{r}{\alpha} \right)' = 1 - 8\pi \left(\rho - \frac{G}{8\pi} \right) r^2. \quad (42)$$

Integrating this equation gives

$$\alpha(r) = \frac{1}{1 - 2\mathcal{M}(r)}, \quad (43)$$

where

$$\mathcal{M}(r) = 4\pi \int_0^r dr' r'^2 \tilde{\rho}(r'), \quad (44)$$

and

$$\tilde{\rho}(r) = \rho(r) - \frac{G(r)}{8\pi}. \quad (45)$$

From (36), we have

$$\frac{\gamma'}{\gamma} = -\frac{2p'}{p + \rho}. \quad (46)$$

By using Eqs.(39b), (43) and (46), we obtain the extended Oppenheimer-Volkoff equation [15,16]:

$$p'(r) = -\frac{\mathcal{M}(r)\rho(r)}{r^2} \left(1 + \frac{p(r)}{\rho(r)}\right) \left(1 + \frac{4\pi r^3 P(r)}{\mathcal{M}(r)}\right) \left(1 - \frac{2\mathcal{M}(r)}{r}\right)^{-1}, \quad (47)$$

where

$$P(r) = 4\pi \left[p(r) + \left(\frac{N(r) - G(r)}{4\pi} \right) \right]. \quad (48)$$

Eq.(47) reduces in the non-relativistic limit to the Newtonian equation for stellar hydrostatic equilibrium. It is the same as the Oppenheimer-Volkoff equation, except that $\rho(r)$ is replaced by $\tilde{\rho}(r)$ in $\mathcal{M}(r)$ and $p(r)$ is replaced by $P(r)$ in the third factor on the right-hand side.

By using (47) we get

$$\frac{\gamma'(r)}{\gamma(r)} = \frac{2}{r^2} [\mathcal{M}(r) + 4\pi r^3 P(r)] \left(1 - \frac{2\mathcal{M}(r)}{r}\right)^{-1}. \quad (49)$$

The solution that has $\gamma(\infty) = 1$ is given by

$$\gamma(r) = \exp \left\{ -2 \int_r^\infty \frac{dr'}{r'^2} [\mathcal{M}(r') + 4\pi r'^3 P(r')] \left[1 - \frac{2\mathcal{M}(r')}{r'} \right] \right\}. \quad (50)$$

These approximate results hold for a weak $f(r)$ field and for moderately relativistic systems such as neutron stars.

Consider now the second approximation scheme that holds for large $f(r)$ with $f(r) \gg r^2$.

We get for the 11/ α , 22/ β , 00/ γ and [23]/ f field equations:

$$\begin{aligned} \frac{f'^2}{2\alpha f^2} - \frac{f''}{\alpha f} + \frac{\alpha' f'}{2f\alpha^2} + \frac{\gamma'^2}{4\alpha\gamma^2} + \frac{\gamma'\alpha'}{4\gamma\alpha^2} - \frac{\gamma''}{2\alpha\gamma} - \frac{5f'^2 r^4}{2\alpha f^4} \\ + \frac{f'' r^4}{\alpha f^3} - \frac{\alpha' f' r^4}{2\alpha^2 f^3} + \frac{\alpha' r^3}{\alpha^2 f^2} + \frac{8f' r^3}{\alpha f^3} - \frac{8r^2}{\alpha f^2} = 4\pi(\rho - p), \end{aligned} \quad (51a)$$

$$\frac{1}{r^2} - \frac{f'}{\alpha f r} - \frac{\gamma' f'}{2\alpha\gamma f} - \frac{\alpha'}{2\alpha^2 r} - \frac{f''}{\alpha f} + \frac{\alpha' f'}{2\alpha^2 f} + \frac{1}{\alpha r^2} + \frac{\gamma'}{2\alpha\gamma r} + \frac{f'^2}{2\alpha f^2} = 4\pi(\rho - p), \quad (51b)$$

$$- \frac{\gamma'\alpha'}{4\alpha^2\gamma} + \frac{\gamma' f'}{2\alpha\gamma f} + \frac{\gamma''}{2\alpha\gamma} - \frac{\gamma'^2}{4\alpha\gamma^2} + \frac{\gamma' r^3}{\alpha\gamma f^2} - \frac{\gamma' f' r^4}{2\alpha\gamma f^3} = 4\pi(\rho + 3p), \quad (51c)$$

$$\begin{aligned} - \frac{\alpha' f' r^4}{2\alpha^2 f^3} - \frac{f'^2 r^4}{\alpha f^4} - \frac{f''}{2\alpha f} + \frac{\alpha' f'}{4\alpha^2 f} - \frac{\gamma' f'}{4\alpha\gamma f} + \frac{3f' r^3}{\alpha f^3} - \frac{\gamma' r^3}{\alpha\gamma f^2} - \frac{4r^2}{\alpha f^2} \\ + \frac{\alpha' r^3}{\alpha^2 f^2} + \frac{f'' r^4}{\alpha f^3} + \frac{\gamma' f' r^4}{2\alpha\gamma f^3} = 4\pi(\rho - p). \end{aligned} \quad (51d)$$

By forming the combination: $11/\alpha + 00/\gamma$, we get

$$\begin{aligned} \frac{f'^2}{2\alpha f^2} - \frac{f''}{\alpha f} + \frac{\alpha' f'}{2\alpha^2 f} - \frac{5f'^2 r^4}{2\alpha f^4} + \frac{f'' r^4}{\alpha f^3} - \frac{\alpha' f' r^4}{2\alpha^2 f^3} + \frac{8f' r^3}{\alpha f^3} - \frac{8r^2}{\alpha f^2} \\ + \frac{\alpha' r^3}{\alpha^2 f^2} + \frac{\gamma'}{\gamma} \left(\frac{f'}{2\alpha f} + \frac{r^3}{\alpha f^2} - \frac{f' r^4}{2\alpha f^3} \right) = 8\pi(\rho - p). \end{aligned} \quad (52)$$

Using (46), we find the hydrostatic equilibrium equation:

$$\begin{aligned} p' = - \left[\frac{\alpha f^3(\rho + p)}{(f' f^2 + 2f r^3 - f' r^4)} \right] \left[\frac{f''}{\alpha f} - \frac{f'^2}{2\alpha f^2} - \frac{\alpha' f'}{2\alpha^2 f} + \frac{5f'^2 r^4}{2\alpha f^4} - \frac{f'' r^4}{\alpha f^3} \right. \\ \left. + \frac{\alpha' f' r^4}{2\alpha^2 f^3} - \frac{8f' r^3}{\alpha f^3} + \frac{8r^2}{\alpha f^2} - \frac{\alpha' r^3}{\alpha^2 f^2} + 8\pi(\rho - p) \right]. \end{aligned} \quad (53)$$

Near $r = 0$, we get

$$p' = - \left[\frac{\alpha f(\rho + p)}{f'} \right] \left[\frac{f''}{\alpha f} - \frac{f'^2}{2\alpha f^2} - \frac{\alpha' f'}{2\alpha^2 f} + 8\pi(\rho - p) \right]. \quad (54)$$

We adopt the equation of state for a gas of highly relativistic particles:

$$p = \frac{1}{3}\rho. \quad (55)$$

Using (55) in (54) gives

$$\rho' = - \left(\frac{4\alpha f \rho}{f'} \right) \left(\frac{f''}{\alpha f} - \frac{f'^2}{2\alpha f^2} - \frac{\alpha' f'}{2\alpha^2 f} + \frac{16\pi}{3}\rho \right) \quad (56)$$

In the limit of large densities, $\rho \rightarrow \infty$, we get

$$\rho' = - \left(\frac{64\pi}{3} \right) \left(\frac{\alpha f}{f'} \right) \rho^2. \quad (57)$$

This equation has the formal solution:

$$\rho(r) = \exp \left[-\frac{64\pi}{3} \int^r dr' \frac{\alpha(r') f(r') \rho(r')}{f'(r')} \right] + \text{const.} \quad (58)$$

Thus, in the limit of large ρ and for $f/f' < 0$, at the center of a dense collapsed object, only repulsive forces occur which increase with increasing density. For either a compact object of high density or a very massive object, we expect that an equilibrium state can be achieved.

VI. SPHERICALLY SYMMETRIC COLLAPSE EQUATIONS FOR DUST

If we adopt the approximation scheme leading to the geodesic equation, Eq.(22), for falling test particles, then we can use a comoving coordinate system with the velocity components:

$$u^0 = 1, \quad u^r = u^\theta = u^\phi = 0,$$

and the time dependent metric in normal Gaussian form:

$$ds^2 = dt^2 - \alpha(r, t) dr^2 - \beta(r, t) (d\theta^2 + \sin^2\theta d\phi^2). \quad (59)$$

In order to simplify the field equations, we must make several approximations. We shall assume that the star collapses as a pressureless dust with $p = 0$ and that the direct coupling term in the source tensor, $K_{[\mu\nu]}$, is small and can be neglected during the collapse. As in the last section, we also set $\Lambda = 0$ and take the approximation $\mu \approx 0$. The field equations now take the form:

$$-\frac{1}{2\alpha} A'' - \frac{1}{8\alpha} [(A')^2 + 4B^2] + \frac{\alpha' A'}{4\alpha^2} + \frac{\ddot{\alpha}}{2\alpha} - \frac{\dot{\alpha}^2}{4\alpha^2} + \frac{\dot{\alpha}\dot{A}}{4\alpha} = 4\pi\rho, \quad (60a)$$

$$\begin{aligned} & \frac{1}{\beta} + \frac{1}{4\beta} \left(\frac{2fB - \beta A'}{\alpha} \right)' + \frac{\alpha'}{8\beta} \left(\frac{2fB - \beta A'}{\alpha^2} \right) \\ & + \frac{1}{4} \frac{B(fA' + 2\beta B)}{\alpha\beta} + \frac{1}{4\beta} (\dot{\beta}\dot{A} + \beta\ddot{A}) - \frac{1}{2\beta} (f\dot{D} + \dot{f}D) + \frac{\dot{A}\dot{\alpha}}{8\alpha} \\ & - \frac{1}{4} \left(\frac{fD\dot{\alpha}}{\alpha\beta} \right) - \frac{D}{4\beta} (f\dot{A} + 2\beta\dot{D}) = 4\pi\rho, \end{aligned} \quad (60b)$$

$$-\frac{1}{2}\ddot{A} - \frac{1}{8}(\dot{A}^2 + 4D^2) + \frac{\dot{\alpha}^2}{4\alpha^2} - \frac{1}{2}\frac{\ddot{\alpha}}{\alpha} = 4\pi\rho, \quad (60c)$$

$$-\frac{1}{2}\dot{A}' + \frac{1}{4}\frac{A'\dot{\alpha}}{\alpha} - \frac{1}{8}A'\dot{A} - \frac{1}{2}BD = 0, \quad (60d)$$

$$\begin{aligned} & -\frac{1}{4f}\left(\frac{fA' + 2\beta B}{\alpha}\right)' - \frac{1}{8}\frac{(fA' + 2\beta B)\alpha'}{\alpha^2 f} + \frac{1}{4}\frac{B(2fB - \beta A')}{\alpha f} \\ & + \frac{1}{8}\frac{(f\dot{A} + 2\beta D)\dot{\alpha}}{\alpha f} + \frac{1}{4f}(f\dot{A} + f\ddot{A} + 2\dot{\beta}D + 2\beta\dot{D}) - \frac{D}{4f}(2fD - \beta\dot{A}) = 4\pi\rho. \end{aligned} \quad (60e)$$

From the conservation law (30), we obtain within our approximation scheme:

$$\dot{\rho} + \rho\left(\frac{\dot{\alpha}}{2\alpha} + \frac{\dot{\beta}}{\beta}\right) = 0.$$

From this result, it follows that

$$\frac{\partial}{\partial t}(\rho\beta\sqrt{\alpha}) = 0. \quad (61)$$

There are two approximate regimes that we can adopt in order to further simplify the set of field equations. In the first one, it is assumed that

$$\beta(r, t) \gg f(r, t), \quad (62)$$

while in the second one, we have

$$f(r, t) \gg \beta(r, t). \quad (63)$$

In the first approximation scheme using (62), we obtain from (60a)–(60e) the equations:

$$-\frac{1}{\alpha}\left[\frac{\beta''}{\beta} - \frac{\beta'^2}{2\beta^2} - \frac{\alpha'\beta'}{2\alpha\beta}\right] + \frac{\ddot{\alpha}}{2\alpha} - \frac{\dot{\alpha}^2}{4\alpha^2} + \frac{\dot{\alpha}\dot{\beta}}{2\alpha\beta} + W = 4\pi\rho, \quad (64a)$$

$$\frac{1}{\beta} - \frac{1}{\alpha}\left(\frac{\beta''}{2\beta} - \frac{\alpha'\beta'}{4\alpha\beta}\right) + \frac{\ddot{\beta}}{2\beta} + \frac{\dot{\alpha}\dot{\beta}}{4\alpha\beta} + X = 4\pi\rho, \quad (64b)$$

$$-\frac{\ddot{\alpha}}{2\alpha} - \frac{\ddot{\beta}}{\beta} + \frac{\dot{\alpha}^2}{4\alpha^2} + \frac{\dot{\beta}^2}{2\beta^2} + Y = 4\pi\rho, \quad (64c)$$

$$-\frac{\dot{\beta}'}{\beta} + \frac{\beta'\dot{\beta}}{2\beta^2} + \frac{\dot{\alpha}\beta'}{2\alpha\beta} + Z = 0, \quad (64d)$$

$$\begin{aligned} & \frac{f\dot{\beta}}{2\beta f} - \frac{\ddot{f}}{2f} + \frac{\beta'^2}{2\alpha\beta^2} - \frac{\dot{\beta}^2}{2\beta^2} + \frac{\alpha'\beta'}{2\alpha^2\beta} - \frac{\beta''}{\alpha\beta} + \frac{\ddot{\beta}}{\beta} - \frac{f'\beta'}{2\alpha\beta f} \\ & - \frac{\dot{\alpha}\dot{f}}{4\alpha f} + \frac{\dot{\alpha}\dot{\beta}}{2\alpha\beta} + \frac{f''}{2\alpha f} - \frac{\alpha'f'}{4\alpha^2 f} = 4\pi\rho. \end{aligned} \quad (64e)$$

Here, we have defined

$$W(r, t) = -\frac{\alpha' \beta' f^2}{2\alpha^2 \beta^3} + \frac{\beta'' f^2}{\alpha \beta^3} - \frac{\dot{\alpha} \dot{\beta} f^2}{2\alpha^2 \beta^3} - \frac{\dot{\alpha} \dot{\beta} f^2}{2\alpha \beta^3} - \frac{5\beta'^2 f^2}{2\alpha \beta^4} + \frac{\dot{\alpha} f \dot{f}}{2\alpha \beta^2} + \frac{\alpha' f f'}{2\alpha^2 \beta^2} - \frac{f f''}{\alpha \beta^2} + \frac{4f f' \beta'}{\alpha \beta^3} - \frac{3f'^2}{2\alpha \beta^2}, \quad (65a)$$

$$X(r, t) = -\frac{\dot{\alpha} \dot{\beta} f^2}{2\alpha \beta^3} - \frac{\alpha' \beta' f^2}{2\alpha^2 \beta^3} + \frac{\beta'' f^2}{\alpha \beta^3} - \frac{\ddot{\beta} f^2}{\beta^3} + \frac{\dot{\alpha} f \dot{f}}{2\alpha \beta^2} + \frac{\dot{\beta}^2 f^2}{\beta^4} - \frac{f f''}{\beta^2 \alpha} - \frac{\beta'^2 f^2}{\alpha \beta^4} + \frac{\alpha' f f'}{2\alpha^2 \beta^2} + \frac{\dot{f}^2}{2\beta^2} + \frac{f \ddot{f}}{\beta^2} - \frac{f'^2}{2\alpha \beta^2} - \frac{3f \dot{f} \dot{\beta}}{2\beta^3} + \frac{3f f' \beta'}{2\alpha \beta^3}, \quad (65b)$$

$$Y(r, t) = \frac{\ddot{\beta} f^2}{\beta^3} - \frac{5\dot{\beta}^2 f^2}{2\beta^4} - \frac{3\dot{f}^2}{2\beta^2} + \frac{4\dot{\beta} f \dot{f}}{\beta^3} - \frac{f \ddot{f}}{\beta^2}, \quad (65c)$$

$$Z(r, t) = \frac{\dot{\beta}' f^2}{\beta^3} - \frac{5\dot{\beta} \beta' f^2}{2\beta^4} - \frac{\dot{\alpha} \beta' f^2}{2\alpha \beta^3} + \frac{2\dot{\beta} f f'}{\beta^3} - \frac{f \dot{f}'}{\beta^2} - \frac{3f' \dot{f}}{2\beta^2} + \frac{\dot{\alpha} f f'}{2\alpha \beta^2} + \frac{2\beta' f \dot{f}}{\beta^3}. \quad (65d)$$

In the second approximation scheme, we have for the $11/\alpha, 22/\beta, 00, (01)$ and $[23]/f$ components:

$$-\frac{5f'^2 \beta^2}{2\alpha f^4} + \frac{f'' \beta^2}{\alpha f^3} - \frac{\dot{\alpha} \dot{\beta} \beta^2}{2\alpha f^3} + \frac{\dot{\alpha} \dot{\beta} \beta}{2\alpha f^2} - \frac{\beta'' \beta}{\alpha f^2} - \frac{3\beta'^2}{2\alpha f^2} + \frac{4f' \beta' \beta}{\alpha f^3} + \frac{\alpha' \beta' \beta}{2\alpha^2 f^2} = 4\pi\rho, \quad (66a)$$

$$\frac{1}{\beta} - \frac{\dot{\alpha} \dot{\beta}}{4\alpha \beta} - \frac{\dot{f}^2}{2f^2} + \frac{\ddot{f}}{f} + \frac{\dot{\alpha} \dot{f}}{2\alpha f} + \frac{\alpha' f'}{2\alpha^2 f} - \frac{\ddot{\beta}}{2\beta} - \frac{f''}{\alpha f} + \frac{\dot{f} \dot{\beta}}{2\beta f} + \frac{\beta''}{2\alpha \beta} - \frac{\alpha' \beta'}{4\alpha^2 \beta} - \frac{\beta' f'}{2\alpha \beta f} + \frac{f'^2}{2\alpha f^2} = 4\pi\rho, \quad (66b)$$

$$-\frac{\ddot{\alpha}}{2\alpha} + \frac{\dot{\alpha}^2}{4\alpha^2} + \frac{\dot{f}^2}{2f^2} - \frac{\ddot{f}}{f} - \frac{5\dot{f}^2 \beta^2}{2f^4} + \frac{\ddot{f} \beta^2}{f^3} - \frac{\ddot{\beta} \beta}{f^2} - \frac{3\dot{\beta}^2}{2f^2} + \frac{4\dot{f} \dot{\beta} \beta}{f^3} = 4\pi\rho, \quad (66c)$$

$$\frac{f' \dot{f}}{2f^2} - \frac{\dot{f}'}{f} + \frac{\dot{\alpha} f'}{2\alpha f} - \frac{5f' \dot{f} \beta^2}{2f^4} + \frac{\dot{f}' \beta^2}{f^3} - \frac{\dot{\alpha} f' \beta^2}{2\alpha f^3} + \frac{2\dot{f} \beta' \beta}{f^3} - \frac{3\dot{\beta} \beta'}{2f^2} + \frac{2\dot{\beta} \beta f'}{f^3} - \frac{\dot{\beta}' \beta}{f^2} + \frac{\dot{\alpha} \beta' \beta}{2\alpha f^2} = 0, \quad (66d)$$

$$-\frac{3\dot{f} \dot{\beta} \beta}{2f^3} + \frac{\dot{\beta}^2}{2f^2} - \frac{\dot{\alpha} \dot{\beta} \beta^2}{2\alpha f^3} + \frac{3f' \beta' \beta}{2\alpha^2 f^3} - \frac{f'^2 \beta^2}{\alpha f^4} - \frac{\beta'^2}{2\alpha f^2} + \frac{\alpha' \beta' \beta}{2\alpha^2 f^2} + \frac{\ddot{f}}{2f} - \frac{f''}{2\alpha f} - \frac{\alpha' f' \beta^2}{2\alpha^2 f^3} - \frac{\ddot{f} \beta^2}{f^3} + \frac{\dot{\alpha} \dot{f}}{4\alpha f} + \frac{f'' \beta^2}{\alpha f^3} + \frac{\dot{f}^2 \beta^2}{f^4} + \frac{\alpha' f'}{4\alpha^2 f} + \frac{\dot{\alpha} \dot{\beta} \beta}{2\alpha f^2} - \frac{\beta'' \beta}{\alpha f^2} + \frac{\ddot{\beta} \beta}{f^2} = 4\pi\rho. \quad (66e)$$

Here, we have used

$$B \approx \frac{f\beta' - \beta f'}{f^2}, \quad D \approx \frac{\dot{\beta}f - \dot{f}\beta}{f^2}. \quad (67)$$

VII. ANALYSIS OF DUST COLLAPSE

We shall simplify our model for collapse even further and assume that the density ρ is independent of position. To analyze the field equations, we shall follow the procedures given by Weinberg, Landau and Lifshitz, and by Misner, Thorne and Wheeler [16–18]. Consider first the approximation scheme determined by the condition (62). It is assumed that a solution can be found by a separation of variables:

$$\alpha(r, t) = h(r)R^2(t), \quad \beta(r, t) = r^2S^2(t). \quad (68)$$

From Eq.(64d), we get

$$\frac{\dot{R}}{R} - \frac{\dot{S}}{S} = \frac{1}{2}Z(r, t)r. \quad (69)$$

If, during the collapse, we assume that $Z(r, t) \approx 0$, then from (69) we find that

$$R(t) \approx S(t).$$

Eqs. (64a) and (64b) now become

$$\frac{h'(r)}{rh^2(r)} + \ddot{R}(t)R(t) + 2\dot{R}^2(t) + R^2(t)W(r, t) = 4\pi R^2(t)\rho(t), \quad (70)$$

$$\frac{1}{r^2} - \frac{1}{r^2h(r)} + \frac{h'(r)}{2rh^2(r)} + \ddot{R}(t)R(t) + 2\dot{R}^2(t) + R^2(t)X(r, t) = 4\pi R^2(t)\rho(t). \quad (71)$$

The metric line-element takes the cosmological Friedmann-Robertson-Walker form:

$$ds^2 = dt^2 - R^2(t) \left[h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (72)$$

Thus, we assume that during the collapse the metric is approximately isotropic and homogeneous. If we assume that $W(r, t) \approx X(r, t) \approx 0$, then we find from Eqs.(70) and (71) the GR solution:

$$h(r) = \frac{1}{1 - kr^2}, \quad (73)$$

where

$$2k = \frac{h'(r)}{rh^2(r)} = \frac{1}{r^2} - \frac{1}{r^2 h(r)} + \frac{h'(r)}{2rh^2(r)},$$

is a constant.

We observe at this point that by means of a Killing vector analysis, it can be proved that a time dependent solution of NGT cannot describe an exact homogeneous and isotropic spacetime, unless $g_{[\mu\nu]}$ is identically zero [19]. Let us expand the metric $g_{(\mu\nu)}$ as

$$g_{(\mu\nu)}(r, t) = g_{(\mu\nu)}^{HI}(r, t) + \delta g_{(\mu\nu)}(r, t), \quad (74)$$

where $g_{(\mu\nu)}^{HI}$ denotes the homogeneous and isotropic solution of $g_{(\mu\nu)}$ and $\delta g_{(\mu\nu)}$ are small quantities which break the maximally symmetric solution with constant Riemannian curvature.

Eqs.(64c) and (70) can now be written:

$$2b(r) + \ddot{R}(t)R(t) + 2\dot{R}^2(t) + R^2(t)W(r, t) = 4\pi R^2(t)\rho(t), \quad (75)$$

$$- \ddot{R}(t)R(t) + \frac{1}{3}R^2(t)Y(t) = \frac{4\pi}{3}R^2(t)\rho(t), \quad (76)$$

where

$$2b(r) = \frac{h'(r)}{rh^2(r)}.$$

Eliminating \ddot{R} by adding (75) and (76), we get

$$\dot{R}^2(t) = -b(r) + \frac{8\pi}{3}\tilde{\rho}(r, t)R^2(t), \quad (77)$$

where

$$\tilde{\rho}(r, t) = \rho(t) - \frac{3}{8\pi}H(r, t) \quad (78)$$

and

$$H(r, t) = \frac{1}{2}[W(r, t) + \frac{1}{3}Y(t)].$$

We shall normalize $R(t)$ so that

$$R(0) = 1,$$

and we define

$$\tilde{\rho}(r, t) = \tilde{\rho}(r, 0)R^{-3}(t).$$

Let us assume that the fluid is at rest at $t = 0$, so that $\dot{R}(0) = 0$. We have

$$b(r) = \frac{8\pi}{3}\tilde{\rho}(r, 0),$$

where

$$\tilde{\rho}(r, 0) = \rho(0) - \frac{3}{8\pi}H(r, 0).$$

We now obtain the equation:

$$\dot{R}^2(t) = -b(r) + \frac{8\pi}{3}\tilde{\rho}(r, 0)R^{-1}(t), \quad (79)$$

which can be written as

$$\dot{R}^2(t) = b(r)[R^{-1}(t) - 1]. \quad (80)$$

This has the same form as the corresponding equation in GR, except that $b(r)$ has the additional contribution from the inhomogeneous, r dependent quantity, $H(r, 0)$, due to the skew fields.

Eq.(80) has the parametric solution:

$$t = \frac{\eta + \sin \eta}{2\sqrt{b}}, \quad (81)$$

$$R = \frac{1}{2}(1 + \cos \eta). \quad (82)$$

This solution reveals that $R(t) = 0$ for $\eta = \pi$ and $t = t_0$ where

$$t_0 = \frac{\pi}{2\sqrt{b(r)}} = \frac{\pi}{2} \left(\frac{3}{8\pi\tilde{\rho}(r,0)} \right)^{1/2}.$$

Thus, as in GR, $\tilde{\rho}(r, t) \rightarrow \infty$ as $R(t) \rightarrow 0$ and the fluid sphere with initial density $\tilde{\rho}(r, 0) > 0$ and $p = 0$ will collapse from rest to a state with infinite proper density in the finite time t_0 , provided that $\tilde{\rho}(r, t) > 0$. But $\tilde{\rho}(r, t)$ need not be positive definite, for it contains second order skew curvature contributions. If, for $R(t)$ near zero, we have $\tilde{\rho}(r, t) < 0$, then the collapse could be stopped even for the approximation $f(r, t) \ll \beta(r, t)$.

However, as we shall see in the following sections, the results for $\beta(r, t) \gg f(r, t)$ are only expected to hold at the initial stage of the collapse for $r \gg 2M$. As the collapse proceeds and the star becomes more dense, we should use the second approximate regime for which the condition (63) holds. As before, we assume that a separable solution is possible:

$$\alpha(r, t) = q(r)R^2(t), \quad \beta(r, t) = r^2R^2(t). \quad (83)$$

Then, equation (66c) becomes

$$\begin{aligned} \frac{\ddot{R}}{R} = & -4\pi\rho - \frac{\ddot{f}}{f} + \frac{\dot{f}^2}{2f^2} - \frac{5\dot{f}^2r^4R^4}{2f^4} + \frac{\ddot{f}r^4R^4}{f^3} \\ & - \frac{8r^4\dot{R}^2R^2}{f^2} - \frac{2r^4R^3\ddot{R}}{f^2} + \frac{8\dot{f}r^4R^3\dot{R}}{f^3}. \end{aligned} \quad (84)$$

We can learn about the behavior of NGT by considering the behavior of (84). Near $R(t) = 0$, Eq.(84) becomes

$$\frac{\ddot{R}}{R} = \frac{\dot{f}^2}{2f^2} - \frac{\ddot{f}}{f} - 4\pi\rho. \quad (85)$$

Let us assume that as the collapse approaches $R(t) = 0$:

$$\frac{\dot{f}^2}{2f^2} - \frac{\ddot{f}}{f} - 4\pi\rho > 0. \quad (86)$$

If for a contracting star we have $\dot{R}/R > 0$, and since by definition $R > 0$, then $\ddot{R} > 0$ and $R(t)$ will not pass through zero during the collapse. Thus, the proper circumference and the proper three-volume of the collapsing star remain finite at the end of the collapse. Moreover, the proper density $\rho(t)$ is finite for the final state of the star. In contrast, in GR we have

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}\rho(0)R^{-3}, \quad (87)$$

from which it follows that the fluid sphere of initial density $\rho(0) > 0$ and zero pressure must collapse from rest in a finite time t_0 to a state of infinite proper energy density.

VIII. EXTERIOR STATIC VACUUM SOLUTION

The metric line element in NGT has the form

$$ds^2 = m_{(\mu\nu)}dx^\mu dx^\nu, \quad (88)$$

where $m_{(\mu\nu)}$ can have three forms in the long-range limit $\mu \approx 0$:

$${}^1m_{(\mu\nu)} = g_{(\mu\nu)}, \quad (89a)$$

$${}^2m^{(\mu\nu)} = g^{(\mu\nu)}, \quad (89b)$$

$${}^3m^{(\mu\nu)} = \frac{2s}{g}(g^{(\mu\nu)})^{-1} - g^{(\mu\nu)}, \quad (89c)$$

where $s = \text{Det}(g_{(\mu\nu)})$ and $g = \text{Det}(g_{\mu\nu})$. The metrics ${}^1m_{(\mu\nu)}$, ${}^2m^{(\mu\nu)}$ and ${}^3m^{(\mu\nu)}$ were obtained from a study of the Cauchy evolution of field equations in Einstein's unified field theory by Maurer-Tison [20,21]. There are three light-cones in NGT, corresponding to the propagation of different zero mass modes. These three metrics can describe the causal Minkowskian light-cone structure of the spacetime in NGT, although it is expected that when $\mu \neq 0$, the light cone structure of the spacetime will be somewhat modified, since the massive $g_{[\mu\nu]}$ field will not propagate causal information along the light cone. The three light cones degenerate to the standard single light cone of special relativity, when NGT is expanded about the Minkowski background metric, and also when it is expanded about a GR background metric. At the Schwarzschild radius, $r = 2M$, the three light cones are regrouped by interchanging their overlapping status [20].

In Born-Infeld non-linear electrodynamics [22], there are two kinds of electric fields, one of which is point-like and singular at $r = 0$, while the other one is finite at $r = 0$; for the

latter the electric current density is spread out over space. Similarly, for the spherically symmetric NGT vacuum solution, the Riemannian geometries associated with the metrics 1m , 2m and 3m are singular at $r = 0$, while the non-Riemannian geometry determined by the fundamental tensor $g_{\mu\nu}$ is finite at $r = 0$. The skew field $g_{[\mu\nu]}$ is like a “medium” which diffuses the spacetime metric and makes the non-Riemannian geometry non-singular. The ratio

$$\epsilon = \frac{R_R}{R_{NR}},$$

where R_R and R_{NR} denote the Riemannian and non-Riemannian scalar curvatures, behaves like a “dielectric constant” in spacetime.

It is not meaningful to ask which geometry is the “correct” one, since the Riemannian and non-Riemannian geometries both play dual roles in the description of spacetime. We shall adopt for convenience the definition of the line element:

$$ds^2 = g_{(\mu\nu)} dx^\mu dx^\nu. \quad (90)$$

We will assume that $M \ll 1/\mu$. It can be shown that the only solution which yields an asymptotically Minkowskian spacetime has $w(r) = 0$ [5,14].

In the case of the long-range approximation of NGT, corresponding to $\mu \approx 0$ in the field equations, the exterior static spherically symmetric solution has the form (Wyman [23]):

$$\gamma_{\text{ext}}(r) = e^\nu, \quad (91a)$$

$$\alpha_{\text{ext}}(r) = \frac{M^2 e^{-\nu} (1 + s^2)}{(\cosh(a\nu) - \cos(b\nu))^2} \left(\frac{d\nu}{dr} \right)^2, \quad (91b)$$

$$\beta_{\text{ext}}(r) = r^2, \quad (91c)$$

$$f_{\text{ext}}(r) = \frac{2M^2 e^{-\nu} [\sinh(a\nu) \sin(b\nu) + s(1 - \cosh(a\nu) \cos(b\nu))]}{(\cosh(a\nu) - \cos(b\nu))^2}, \quad (91d)$$

where

$$a = \sqrt{\frac{\sqrt{1 + s^2} + 1}{2}} \quad \text{and} \quad b = \sqrt{\frac{\sqrt{1 + s^2} - 1}{2}}.$$

M is identified with the mass and s is a dimensionless constant which is different for different bodies and is related to the strength of the coupling of matter to the skew field $g_{[\mu\nu]}$.

The function $\nu(r)$ is determined by the relation:

$$e^\nu [\cosh(a\nu) - \cos(b\nu)]^2 \frac{r^2}{2M^2} = \cosh(a\nu) \cos(b\nu) - 1 + s \sinh(a\nu) \sin(b\nu). \quad (92)$$

Two coordinate systems can be used to analyze the Wyman solution, one of which is the standard spherically symmetric coordinates: $x^1 = r, x^2 = \theta, x^3 = \phi, x^0 = t$. Another useful coordinate system uses the coordinates: $x^1 = \nu, x^2 = \theta, x^3 = \phi, x^0 = t$, where

$$\alpha_{\text{ext}}(\nu) = \frac{M^2 e^{-\nu} (1 + s^2)}{[\cosh(a\nu) - \cos(b\nu)]^2}, \quad (93)$$

$$\beta_{\text{ext}}(\nu) = \frac{2M^2 [\cosh(a\nu) \cos(b\nu) - 1 + s \sinh(a\nu) \sin(b\nu)]}{e^\nu [\cosh(a\nu) - \cos(b\nu)]^2}, \quad (94)$$

and with $\gamma_{\text{ext}}(\nu)$ and $f_{\text{ext}}(\nu)$ given as above.

We must choose a particular branch of a solution of Eq.(92). This choice is made by picking the branch that will yield the positive-mass Schwarzschild solution as a limit for $r \rightarrow \infty$. This branch begins at $\nu = 0$ and extends towards negative ν . In such a coordinate system, the asymptotic weak-field region is at $\nu = 0$ ($r = \infty$), while the “origin” occurs at ν_0 defined by $\beta(\nu_0) = 0$. The particular value of ν_0 depends on the value of s , and for $s = 1$, we find numerically that $\nu_0 \approx -5.1667$.

For $r \gg M$ and $|s| < 1$, the metric functions in conventional spherical coordinates are approximated by (for $\mu \neq 0$) [6,7]:

$$\gamma_{\text{ext}}(r) \approx 1 - \frac{2M}{r}, \quad (95a)$$

$$\alpha_{\text{ext}}(r) \approx \left(1 - \frac{2M}{r}\right)^{-1}, \quad (95b)$$

$$f_{\text{ext}}(r) \approx \frac{sM^2}{3} e^{-\mu r} (1 + \mu r). \quad (95c)$$

We see that for large r , the solutions for γ_{ext} and α_{ext} have asymptotically the same form as the Schwarzschild vacuum solution in GR, and $f_{\text{ext}}(r)$ vanishes exponentially fast as $r \rightarrow \infty$.

Near $r = 0$ we can develop expansions where $r/M < 1$ and $0 < |s| < 1$. The leading terms are

$$\gamma_{\text{ext}}(r) = \gamma_0 + \frac{\gamma_0(1 + \mathcal{O}(s^2))}{2|s|} \left(\frac{r}{M}\right)^2 + \mathcal{O}\left(\left(\frac{r}{M}\right)^4\right), \quad (96)$$

$$\alpha_{\text{ext}}(r) = \frac{4\gamma_0(1 + \mathcal{O}(s^2))}{s^2} \left(\frac{r}{M}\right)^2 + \mathcal{O}\left(\left(\frac{r}{M}\right)^4\right), \quad (97)$$

$$f_{\text{ext}}(r) = M^2 \left(4 - \frac{|s|\pi}{2} + s|s| + \mathcal{O}(s^3)\right) + \frac{|s| + s^2\pi/8 + \mathcal{O}(s^3)}{4} r^2 + \mathcal{O}(r^4), \quad (98)$$

where

$$\gamma_0 = \exp\left(-\frac{\pi + 2s}{|s|} + \mathcal{O}(s)\right) \dots$$

These solutions clearly illustrate *the non-analytic nature of the limit $s \rightarrow 0$ in the strong gravitational field regime $0 < r \leq 2M$* [6,7,24]. Thus, the Wyman solution cannot be analytically continued to the Schwarzschild solution of GR for arbitrarily small values of the parameter s in the region $0 < r \leq 2M$. This means that the expansion about a fixed GR background, in the approximation of linear $g_{[\mu\nu]}$, *cannot be considered valid in the strong gravitational regime: $0 < r \leq 2M$* . This fact will play an important role in our derivation of a solution to the collapse problem in NGT.

To see that the non-analytic behavior in s of the Wyman solution for $0 < r \leq 2M$ is not a coordinate dependent result, we can use the coordinate invariant norm:

$$\sqrt{{}^{(t)}\xi^\mu {}^{(t)}\xi_\mu} = \gamma,$$

where ${}^{(t)}\xi_\mu$ is the timelike Killing vector at spatial infinity:

$${}^{(t)}\xi_\mu = (\gamma, 0, 0, 0).$$

Since γ never vanishes throughout the spacetime, there are no event horizons and the solution is not analytic to the Schwarzschild solution, for $0 < r \leq 2M$ as $s \rightarrow 0$, in any coordinate frame. The redshift is finite for $s \neq 0$ and the maximum redshift determined between $r = 0$ and $r = \infty$ is given by

$$z_{\text{max}} = \frac{1}{\sqrt{\gamma}} - 1.$$

The singularity caused by the vanishing of $\alpha(r)$ at $r = 0$ is a coordinate singularity, which can be removed by transforming to another coordinate frame of reference [6,7].

The non-Riemannian geometry is non-singular in the range $0 \leq r < \infty$, since all the nonsymmetric curvature tensor invariants are finite in this range of r . For example, from the non-Riemmanian curvature tensor:

$$R^\lambda_{\mu\nu\rho}(\Gamma) = \Gamma^\lambda_{\mu\nu,\rho} - \Gamma^\lambda_{\mu\rho,\nu} - \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\rho} + \Gamma^\lambda_{\alpha\rho}\Gamma^\alpha_{\mu\nu},$$

we find that the Kretschmann invariant:

$$K = R^{\lambda\mu\nu\rho}(\Gamma)R_{\lambda\mu\nu\rho}(\Gamma),$$

where

$$R_{\lambda\mu\nu\rho}(\Gamma) = g_{\lambda\sigma}R^\sigma_{\mu\nu\rho}(\Gamma)$$

is finite. On the other hand, the curvature invariants formed from the Riemann-Christoffel curvature tensor $B^\lambda_{\mu\nu\rho}$, defined by

$$B^\lambda_{\mu\nu\rho} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}_{,\rho} - \left\{ \begin{matrix} \lambda \\ \mu\rho \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \lambda \\ \alpha\nu \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \mu\rho \end{matrix} \right\} + \left\{ \begin{matrix} \lambda \\ \alpha\rho \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}$$

are singular like $\sim M^4/r^8$ near $r = 0$ [3,25]. The Riemann-Christoffel curvature tensors built out of the two metrics 2m and 3m will also have this singular type of behavior near $r = 0$.

In the ν coordinate representation, $\gamma_{\text{ext}}(\nu)$ vanishes at the point $\nu = -\infty$, corresponding to $r^* = (r^4 + f^2)^{1/4} = 0$, which is a point outside the physical causal spacetime. Moreover, there is a singularity in the non-Riemannian curvature invariants at the unphysical point $\nu = -\infty$ [25].

Let us consider the exterior static spherically symmetric line element given by the metric ${}^1m_{(\mu\nu)} = g_{(\mu\nu)}$:

$$ds^2 = \gamma_{\text{ext}}(\nu)dt^2 - \alpha_{\text{ext}}(\nu)d\nu^2 - \beta_{\text{ext}}(\nu)(d\theta^2 + \sin^2\theta d\phi^2). \quad (99)$$

We see that $\gamma_{\text{ext}}(\nu)$ does not vanish in the range: $-\infty < \nu \leq 0$. Thus, all the points of spacetime are described by timelike Killing vector fields in the region $-\infty < \nu \leq 0$; they

can be causally connected and all particles can be at rest. This corresponds to the fact that there are no trapped surfaces (event horizons) in this range of ν . There are no event horizons in the r -coordinate frame in the region: $0 < r \leq \infty$.

Test particles which follow radial trajectories according to the path equation [3]:

$$\frac{du^\mu}{ds} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta = 0$$

or test particles that follow radial trajectories according to the geodesic equation, Eq.(22), are not stopped at $r = 0$ but can continue through into the unphysical vacuum manifold for $r < 0$ [13]. However, we shall see that in the collapse of physical bodies, it is unlikely that matter can collapse through the point $r = 0$ and we conjecture that this is never possible.

IX. MATCHING OF INTERIOR AND EXTERIOR SOLUTIONS

We must consider now the matching of the metric outside the star with the one in the interior of the star. Birkhoff's theorem does not hold for the spherically symmetric vacuum solution of NGT, which makes the matching of solutions a more difficult task to solve than in GR. It can be proved for $\mu \neq 0$ that no monopole radiation can escape to asymptotic infinity [5].

The exterior metric outside the star can be expressed in terms of the coordinates, $\bar{r}, \bar{\theta}, \bar{\phi}, \bar{t}$ in the form:

$$ds^2 = C(\bar{r}, \bar{t})d\bar{t}^2 - D(\bar{r}, \bar{t})d\bar{r}^2 - 2E(\bar{r}, \bar{t})d\bar{r}d\bar{t} - \bar{r}^2(d\bar{\theta}^2 + \sin^2 \bar{\theta}d\bar{\phi}^2). \quad (100)$$

We remove E by defining the new time variable:

$$d\bar{t}' = \epsilon(\bar{r}, \bar{t})[C(\bar{r}, \bar{t})d\bar{t} - E(\bar{r}, \bar{t})d\bar{r}]. \quad (101)$$

The line element now takes the form [16]:

$$ds^2 = \gamma_{\text{ext}}(\bar{r}, \bar{t})d\bar{t}'^2 - \alpha_{\text{ext}}(\bar{r}, \bar{t})d\bar{r}^2 - \bar{r}^2(d\bar{\theta}^2 + \sin^2 \bar{\theta}d\bar{\phi}^2), \quad (102)$$

where

$$\gamma_{\text{ext}}(\bar{r}, \bar{t}) = \epsilon^{-2}(\bar{r}, \bar{t}) C^{-1}(\bar{r}, \bar{t}), \quad (103)$$

$$\alpha_{\text{ext}}(\bar{r}, \bar{t}) = D(\bar{r}, \bar{t}) + C^{-1}(\bar{r}, \bar{t}) E^2(\bar{r}, \bar{t}). \quad (104)$$

Let us first consider the approximation scheme in which Eq.(62) is valid. This approximation is expected to hold for the initial phase of the collapse when f is small. In order to match the solutions at the surface of the star, we must convert the interior solution with the metric (72) into “standard” coordinates. We shall assume that in the initial phase of the collapse, the interior metric is approximately determined by the solution Eq.(73). We choose [16]

$$\bar{r} = rR(t), \quad \bar{\theta} = \theta, \quad \bar{\phi} = \phi, \quad (105)$$

and use an integrating factor which yields

$$\bar{t} = \left(\frac{1 - kr_0^2}{k} \right) \int_{Q(r,t)}^1 \frac{dR}{\left(1 - \frac{kr_0^2}{R^2} \right)} \left(\frac{R}{1 - R} \right)^{1/2}, \quad (106)$$

where

$$Q(r, t) = 1 - \left(\frac{1 - kr^2}{1 - kr_0^2} \right)^{1/2} (1 - R(t)).$$

Here the constant r_0 is set equal to the radius of the star in comoving coordinates. We now get

$$\gamma(\bar{r}, \bar{t}) = \frac{R}{Q} \left(\frac{1 - kr^2}{1 - kr_0^2} \right)^{1/2}, \quad (107)$$

$$\alpha(\bar{r}, \bar{t}) = \left(1 - \frac{kr^2}{R} \right)^{-1}, \quad (108)$$

where Q and R are functions of \bar{r} and \bar{t} . At the radius of the star, we have

$$\bar{r} = \bar{r}_0 = r_0 R(t) \quad (109)$$

$$\bar{t} = \left(\frac{1 - kr_0^2}{k} \right)^{1/2} \int_{R(t)}^1 \frac{dR}{\left(1 - \frac{kr_0^2}{R^2} \right)} \left(\frac{R}{1 - R} \right)^{1/2}, \quad (110)$$

$$\gamma(\bar{r}_0, \bar{t}) = 1 - \frac{kr_0^2}{R(t)}, \quad (111)$$

$$\alpha(\bar{r}_0, \bar{t}) = \left(1 - \frac{kr_0^2}{R(t)} \right)^{-1}. \quad (112)$$

The exterior and interior solutions match at $\bar{r} = r_0 R(t)$ if we have

$$\gamma(\bar{r}_0, \bar{t}) = \gamma_{\text{ext}}(\bar{r}_0, \bar{t}), \quad (113a)$$

$$\alpha(\bar{r}_0, \bar{t}) = \alpha_{\text{ext}}(\bar{r}_0, \bar{t}), \quad (113b)$$

$$f(\bar{r}_0, \bar{t}) = f_{\text{ext}}(\bar{r}_0, \bar{t}). \quad (113c)$$

We shall now expand the exterior time dependent solution as

$$\gamma_{\text{ext}}(\bar{r}, \bar{t}) = \gamma_{\text{ext}}(\bar{r}) + \delta\gamma_{\text{ext}}(\bar{r}, \bar{t}), \quad (114a)$$

$$\alpha_{\text{ext}}(\bar{r}, \bar{t}) = \alpha_{\text{ext}}(\bar{r}) + \delta\alpha_{\text{ext}}(\bar{r}, \bar{t}), \quad (114b)$$

$$f_{\text{ext}}(\bar{r}, \bar{t}) = f_{\text{ext}}(\bar{r}) + \delta f_{\text{ext}}(\bar{r}, \bar{t}). \quad (114c)$$

We assume that $\delta\gamma_{\text{ext}}$, $\delta\alpha_{\text{ext}}$ and δf_{ext} are small quantities that can be neglected during the collapse (quasi-static approximation). The $\alpha_{\text{ext}}(\bar{r})$, $\gamma_{\text{ext}}(\bar{r})$ and $f_{\text{ext}}(\bar{r})$ are determined by the unique static solution of the NGT vacuum field equations.

For large values of the star's surface radius, $r_0 \gg 2M$, we have

$$\alpha_{\text{ext}}(r_0) = \left(1 - \frac{2M}{r_0}\right)^{-1}, \quad (115)$$

$$\gamma_{\text{ext}}(r_0) = 1 - \frac{2M}{r_0}, \quad (116)$$

$$f_{\text{ext}}(r_0) = \frac{sM^2}{3} \exp(-r_0/a)(1 + r_0/a). \quad (117)$$

If, as we have been assuming, $r_0 \ll a$, then we get from (117):

$$f_{\text{ext}}(r_0) = \frac{sM^2}{3}. \quad (118)$$

Let us assume that s is related to the strength of the skew field coupling to matter by the expression:

$$s = \frac{g}{N_B^\beta}, \quad (119)$$

where g is a coupling constant, N_B is the baryon number of the star and β is some dimensionless constant. We expect that for a non-zero coupling of the skew field to matter there should exist a non-vanishing static exterior solution for f .

The interior and exterior solutions fit for large r_0 when $\bar{r} = r_0 R(t)$ if:

$$kr_0^2 = \frac{2M}{r_0}. \quad (120)$$

This yields for large r_0 the expression for the total mass of the star:

$$M = \frac{4\pi}{3} \rho(0) r_0^3. \quad (121)$$

For $r_0 R(t) \rightarrow 2M$, we must switch to the approximation scheme based on the condition (63), since the quasi-static Schwarzschild solution fails to be a solution of the NGT field equations for $r_0 R \leq 2M$ [24], provided we make the reasonable assumption that the limit to the static solution from small time dependent perturbations is smooth. We assume as before that the solution can be expressed as a separation of variables, as in Eq.(83). Then the line element takes the normal Gaussian form:

$$ds^2 = dt^2 - R^2(t)[q(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (122)$$

To express this line element in standard form, we must use the integration factor:

$$\bar{t}' = \int dR \mathcal{F}(R, r_0), \quad (123)$$

where \mathcal{F} is a function that is chosen to remove the cross-term $drdt$. The matching of the exterior and interior solutions is achieved by use of the conditions (113a)-(113c).

Because there are no trapped surfaces (event horizons) in the exterior Wyman solution in the range $0 < \bar{r} \leq \infty$, we have

$$\gamma_{\text{ext}}(2M) \neq 0, \quad \alpha_{\text{ext}}(2M) < \infty, \quad f_{\text{ext}}(2M) \neq 0. \quad (124)$$

From this we can deduce that the matching of the interior and the exterior solutions will *not produce a black hole event horizon in the final stage of collapse.*

A light signal emitted in a radial direction at \bar{t} will have $d\bar{r}/d\bar{t}$ determined by $ds^2 = 0$ where

$$ds^2 = \gamma(\bar{r}, \bar{t})d\bar{t}^2 - \alpha(\bar{r}, \bar{t})d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

The light signal will be detected at a distant point \bar{r} at a time:

$$\bar{t}' = \bar{t} + \int_{r_0 R(t)}^{\bar{r}'} d\bar{r} \left[\frac{\alpha_{\text{ext}}(\bar{r})}{\gamma_{\text{ext}}(\bar{r})} \right]^{1/2}.$$

Because of the bounds in (124), we find that $\bar{t}' < \infty$ and the collapse to the Schwarzschild radius can occur in a *finite* time.

In GR, we have

$$\bar{t}' \sim -2kr_0^3 \ln \left[1 - \frac{kr_0^2}{R(t)} \right] + \text{const},$$

so that as $r_0 R \rightarrow 2M = kr_0^3$, it takes an infinite amount of time for the star to collapse to the Schwarzschild radius.

For large r_0 , we have

$$\frac{kr^2}{R(t)} \ll 1$$

and the redshift is given by

$$z \sim r_0 \sqrt{k} \left(\frac{1 - R(t)}{R(t)} \right)^{1/2} \sim r_0 \sqrt{k} \left(1 - R(\bar{t}' - \bar{r}') (R(\bar{t}' - \bar{r}')) \right)^{1/2},$$

which is asymptotically the same as the result in GR.

As $r_0 \rightarrow 2M$, the redshift is determined by the formula:

$$z \equiv \frac{\lambda' - \lambda_0}{\lambda_0} = \frac{d\bar{t}'}{dt} - r_0 \dot{R}(t) \left[\frac{\alpha_{\text{ext}}(r_0 R)}{\gamma_{\text{ext}}(r_0 R)} \right]^{1/2} - 1. \quad (125)$$

The red shift z seen by an observer is zero when the collapse is observed to begin, increases slowly during the collapse but satisfies the bound $z < \infty$, because Eq.(124) holds for $0 < r_0 R(t) < \infty$.

In GR, the redshift becomes for $r_0 R(t) \rightarrow 2M$:

$$z \sim 2 \left(1 - \frac{kr_0^2}{R(t)} \right)^{-1} \sim \exp \left(\frac{\bar{t}'}{2kr_0^3} \right), \quad (126)$$

and the red shift becomes infinitely large as the collapsing star approaches the black hole event horizon.

Since a black hole event horizon does not form in the collapse of a star, in NGT, if we formulate the collapse problem with our matching conditions, the star is not cut off from the rest of the universe, and it can continue to emit all forms of radiation. The collapsed star is expected to form a dense static object that could be a strong source of gravitational radiation and other forms of radiation.

X. CONCLUDING REMARKS

We have been able to find an approximate treatment, in NGT, of the stellar equilibrium problem. We found that it is possible to realize a static equilibrium state for a massive compact object, due to the strong repulsive forces at the center of the object produced by the skew fields. We also formulated an approximate treatment of the gravitational collapse of a star considered as a spherically symmetric, pressureless dust sphere. By matching the exterior vacuum solution to the interior solution, we have shown that black holes are not expected to form during the collapse, since the exterior solution for small time dependent perturbations can approximate the static Wyman solution, which does not possess any trapped surfaces. The collapse can be stopped before the center of the star is reached. To completely solve the problem of the collapse of a star, in NGT, we must find a solution of the full NGT field equations, preferably for $\mu \neq 0$ and including a suitable $K^{[\mu\nu]}$ source contribution. Such a solution is expected to be found only by using numerical methods to solve the field equations.

It is important to stress that the collapse of a star, in NGT, must be stopped before $R(t) = 0$ is reached, since the spacetime becomes unphysical for $R(t) < 0$. Because of the absence of black hole event horizons, such unphysical behavior would be “naked” and it would destroy the physical Cauchy data and make NGT a non-viable classical theory of gravitation. Of course, it is possible that we would have to discover a quantum theory of gravity to fully comprehend the collapse to small distances. But since such a theory is not presently available, it would be more satisfactory to be able to settle the issue of small

distance collapse within the classical regime of NGT.

It has been conjectured by Burko and Ori [10] that black holes should be anticipated in the gravitational collapse of stars in NGT. Their analysis relied entirely on the use of an expansion of the NGT time dependent field equations about a GR background to first order in small $g_{[\mu\nu]}$. Specifically, this background was chosen to be the static spherically symmetric Schwarzschild solution. Of course, with this assumption a black hole is expected to form, because of the existence of a no-hair theorem for a skew symmetric potential coupled to a Schwarzschild metric [11], and the existence of trapped surfaces in the Schwarzschild solution.

Since Burko and Ori demanded that f be small throughout the collapse of a star, they used a quasi-static exterior metric that can be closely approximated by the static Schwarzschild solution for small enough f . This metric *cannot* be a solution of the NGT vacuum field equations for $0 < r \leq 2M$ [24]. The exact static spherically symmetric vacuum solution, in NGT, is the Wyman solution which does not have event horizons for $0 < r \leq \infty$. It is *not* analytic to the Schwarzschild solution for arbitrarily small values of the parameter s in the range $0 < r \leq 2M$. From this we conclude that a quasi-static exterior metric is expected to be the Wyman metric plus a small time dependent part, which can be valid for suitably chosen initial value data with small f , before the onset of collapse. When the interior solution is matched to this exterior solution, black holes are not expected to form during collapse. This invalidates the claim made by Burko and Ori that black holes can be anticipated in the collapse of a star in NGT.

It is incorrect to claim any definitive results for non-linear gravity theories, such as GR and NGT, on the basis of the linear approximation. One would not use the linear approximation in GR to solve the collapse problem, because if we expand the metric about Minkowski space, then for $r = 2M$, the perturbative expansion fails to be valid, since the metric perturbation is of order 1. Similarly, the Burko-Ori quasi-static expansion of $g_{\mu\nu}$ about the Schwarzschild background fails to hold at $r = 2M$, because $g_{[\mu\nu]}$ becomes larger than unity and the linear approximation breaks down. Given that a generic source exists

that generates a non-zero $g_{[\mu\nu]}$, a static solution for f is produced. Then, we know from the exact Wyman solution that f becomes large near $r = 2M$, and the linear equation for f fails to describe correctly the physical collapse of a star. Assuming that f is just a radiating wave without a static part during the collapse is not a physically realistic treatment of the problem in NGT. Burko and Ori incorrectly assumed that by adding matter and a generic coupling of matter to the skew fields for gravitational collapse would not change their conclusions.

The NGT violates the strong equivalence principle and, therefore, it is expected that a freely falling observer (the weak equivalence principle is not violated in the new massive version of NGT) could perform experiments to detect the formation of an event horizon. This is not possible in GR. This effect would only show itself in higher orders of $g_{[\mu\nu]}$, and would have important consequences for the physics of collapse in NGT.

Actual stars would collapse more slowly than in the model which we have studied because of the effect of the pressure of radiation, of matter and of rotation.

Since the final collapsed object is expected to be a massive compact star without an event horizon, radiation of all forms can be emitted by the surface of the star. Of course, if the red shift emitted from the surface of the star is too large (but never infinite), then in practice only small amounts of thermal and gravitational radiation will escape. There would be no Hawking radiation emitted, for such radiation is associated specifically with a rigorous black hole event horizon. Therefore, the problem of information loss associated with the quantum mechanical aspects of a black hole would be eliminated.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada. I thank M. A. Clayton, N. J. Cornish, L. Demopoulos, J. Légaré, M. Reisenberger and P. Savaria for helpful discussions.

APPENDIX A: THE TIME DEPENDENT Γ -CONNECTIONS

The NGT compatibility equation is given by

$$g_{\lambda\nu,\eta} - g_{\rho\nu}\Gamma_{\lambda\eta}^{\rho} - g_{\lambda\rho}\Gamma_{\eta\nu}^{\rho} = \frac{1}{6}g^{(\mu\rho)}(g_{\rho\nu}g_{\lambda\eta} - g_{\eta\nu}g_{\lambda\rho} - g_{\lambda\nu}g_{[\rho\eta]})W_{\mu}, \quad (\text{A1})$$

where W_{μ} is determined from (23). For the spherically symmetric system, when $w(r, t) = g_{[01]}(r, t) = 0$, it follows that $W_{\mu} = 0$ and the compatibility equation reads:

$$g_{\lambda\nu,\eta} - g_{\rho\nu}\Gamma_{\lambda\eta}^{\rho} - g_{\lambda\rho}\Gamma_{\eta\nu}^{\rho} = 0. \quad (\text{A2})$$

The non-vanishing components of the Γ -connections are:

$$\Gamma_{11}^1 = \frac{\alpha'}{2\alpha}, \quad (\text{A3})$$

$$\Gamma_{(10)}^1 = \frac{\dot{\alpha}}{2\alpha}, \quad (\text{A4})$$

$$\Gamma_{22}^1 = \Gamma_{33}^1 \text{cosec}^2\theta = \frac{1}{2\alpha}\left(fB - \frac{1}{2}\beta A'\right), \quad (\text{A5})$$

$$\Gamma_{00}^1 = \frac{\gamma'}{2\alpha}, \quad (\text{A6})$$

$$\Gamma_{(12)}^2 = \Gamma_{(13)}^3 = \frac{1}{4}A', \quad (\text{A7})$$

$$\Gamma_{(20)}^2 = \Gamma_{(30)}^3 = \frac{1}{4}\dot{A}, \quad (\text{A8})$$

$$\Gamma_{33}^2 = -\sin\theta \cos\theta, \quad (\text{A9})$$

$$\Gamma_{(23)}^3 = \cot\theta, \quad (\text{A10})$$

$$\Gamma_{(11)}^0 = \frac{\dot{\alpha}}{2\gamma}, \quad (\text{A11})$$

$$\Gamma_{(10)}^0 = \frac{\gamma'}{2\gamma}, \quad (\text{A12})$$

$$\Gamma_{22}^0 = \Gamma_{33}^0 \text{cosec}^2\theta = -\frac{1}{2\gamma}\left(fD - \frac{1}{2}\beta\dot{A}\right), \quad (\text{A13})$$

$$\Gamma_{00}^0 = \frac{\dot{\gamma}}{2\gamma}, \quad (\text{A14})$$

$$\Gamma_{[23]}^1 = \frac{\sin\theta}{2\alpha}\left(\frac{1}{2}fA' + \beta B\right), \quad (\text{A15})$$

$$\Gamma_{[13]}^2 = -\Gamma_{[12]}^3 \sin^2\theta = \frac{1}{2}B \sin\theta, \quad (\text{A16})$$

$$\Gamma_{[30]}^2 = -\Gamma_{[20]}^3 \sin^2\theta = -\frac{1}{2}D \sin\theta, \quad (\text{A17})$$

$$\Gamma_{[23]}^0 = -\frac{\sin\theta}{2\gamma}\left(\frac{1}{2}f\dot{A} + \beta D\right), \quad (\text{A18})$$

where A, B and D are given by Eqs.(34a), (34b) and (34c).

REFERENCES

- [1] J.W. Moffat, Phys. Lett. **B355**, 447 (1995).
- [2] J.W. Moffat, J. Math. Phys. **36**, 3722 (1995); *erratum*, J. Math. Phys., to be published.
- [3] J. W. Moffat, J. Math. Phys. **36**, 5897 1995.
- [4] J. Légaré and J. W. Moffat, Gen. Rel. Grav. **27**, 761 (1995).
- [5] M. A. Clayton, J. Math. Phys., to be published.
- [6] N. J. Cornish and J. W. Moffat, Phys. Lett. B **336**, 337 (1994).
- [7] N. J. Cornish and J. W. Moffat, J. Math. Phys. **35**, 6628 (1994).
- [8] S. Hawking, Phys. Rev. D **14**, 2460 (1976); Commun. Math. Phys. **87**, 395 (1982).
- [9] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1995).
- [10] L. M. Burko and A. Ori, Phys. Rev. Lett. **75**, 2455 (1995).
- [11] N. J. Cornish and J. W. Moffat, University of Toronto preprint UTPT-95-14, gr-qc/9506055, 1995.
- [12] A. Papapetrou, Proc. R. Soc. London **A209**, 248 (1951); *Lectures in General Relativity* (Reidel, Dordrecht- Holland/Boston - U.S.A., 1974), p.149.
- [13] J. Légaré and J. W. Moffat, University of Toronto preprint, UTPT-95-19. gr-qc/9509035.
- [14] N. J. Cornish, University of Toronto preprint, UTPT-94-37, 1994 (unpublished).
- [15] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).
- [16] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
- [17] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, 1975).

- [18] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, 1973).
- [19] J. W. Moffat, in *Gravitation - A Banff Summer School*. Proceedings of the Banff Summer School eds. R. B. Mann and P. Wesson (World Scientific, 1991) p. 523.
- [20] F. Maurer-Tison, C. R. Acad. Sc. **242**, 1127 (1956); Ann. Scient. Éc. Norm. Sup. **76**, 185 (1959).
- [21] M. A. Tonnelat, *Einstein's Theory of Unified Fields* (Gordon and Breach Science Publishers, translated from the French by R. Akerbib, 1982).
- [22] M. Born, Proc. R. Soc., London, **143A**, 410 (1934); M. Born and L. Infeld, Proc. R. Soc., London, **144A**, 425 (1934).
- [23] M. Wyman, Can. J. Math. **2**, 427 (1950); W. B. Bonnor, Proc. Roy. Soc. A **209**, 353 (1951).
- [24] J. W. Moffat and I. Yu. Sokolov, University of Toronto preprint, UTPT-95-21, 1995. astro-ph/9510068.
- [25] N. J. Cornish, in preparation.